

# ANALYTIC POISSON BRACKETS ON RATIONAL FUNCTIONS ON THE RIEMANN SPHERE AND THEIR APPLICATIONS.

K.L. VANINSKY

ABSTRACT. We consider a hierarchy of Poisson structures defined on rational functions on the Riemann sphere. This hierarchy is originated in the theory of the integrable Camassa-Holm equation associated with the Krein's string spectral problem. Previously the proof of Jacobi identity was obtained by reducing the bracket to canonical Darboux coordinates. The main result of this note is a direct proof of the Jacobi identity. It turns out that the direct proof of the Jacobi identity is far from trivial. We also give an example of another hierarchy of Poisson brackets and construct Darboux coordinates for it.

## CONTENTS

1. Introduction	1
2. The first proof of Jacobi identity.	3
3. The second proof of Jacobi identity.	5
4. The Poisson bracket in terms of the Weyl function.	9
5. Another Hierarchy of Poisson brackets.	10
References	14

## 1. INTRODUCTION

Consider a space of rational functions  $\mathbf{Rat}_N$  on the Riemann sphere  $\mathbb{CP}$  which can be represented as

$$w(z) = \sum_{k=1}^N \frac{\rho_k}{z_k - z} = -\frac{q(z)}{p(z)},$$

Note that the space  $\mathbf{Rat}_N$  has complex dimension  $2N$ . When the parameters  $z_k$  and  $\rho_k > 0$  are real than  $w(z)$  maps the upper half-plane into itself.

A symplectic structure on such space of functions was introduced by Atiyah and Hitchin in [1] as

$$\sum_{k=1}^N \frac{dq(z_k)}{q(z_k)} \wedge dp(z_k).$$

1

The corresponding Poisson structure is given by the formula

$$\{w(p), w(q)\} = \frac{(w(p) - w(q))^2}{p - q}. \quad (1.1)$$

This form was found in a remarkable paper of Faybusovich and Gekhtman, [2]. For the Atiyah-Hitchin bracket 1.1 it was shown in [5] that it corresponds to the main Poisson bracket for the Camassa-Holm equation written in terms of the Weyl function, [6], of the associated Krein's string spectral problem .

Faybusovich and Gekhtman also found higher brackets of the infinite hierarchy with 1.1 being the first bracket. In our paper KV and Gekhtman, [3], we found an algebraic-geometrical representation of all these brackets. In [3] it was explain that these brackets produce hierarchy of Poisson brackets of the Camassa-Holm equation. To introduce a formula for the hierarchy we consider a differential  $\alpha_{pq}^f$  on the Riemann sphere  $\mathbb{CP}$  which depends on the entire function  $f(z)$  and two points  $p$  and  $q$

$$\alpha_{pq}^f = \frac{\epsilon_{pq}(z)}{p - q} \times f(z)w(z) (w(p) - w(q)),$$

where

$$\epsilon_{pq}(z) = \frac{1}{2\pi i} \left[ \frac{1}{z - p} - \frac{1}{z - q} \right] dz$$

is the standard differential Abelian differential of the third kind with residues  $\pm 1$  at the points  $p$  and  $q$ . It can be written in the form

$$\begin{aligned} \alpha_{pq}^f &= \epsilon_{pq}^\circ(z) \times f(z)w(z) (w(p) - w(q)) \\ &= \frac{1}{2\pi i} \frac{dz}{(z - p)(z - q)} \times f(z)w(z) (w(p) - w(q)) \end{aligned}$$

that is used later. The analytic Poisson brackets are defined on  $\mathbf{Rat}_N$  by the formula, see [3]:

$$\{w(p), w(q)\}^f = \sum_{k=1}^N \int_{\hat{O}_k} \alpha_{pq}^f, \quad (1.2)$$

where the circles  $O_k$  are traversed counter clockwise and surround points  $z_k$ . When  $f(z) = 1$  we obtain 1.1 from 1.2. Another closed formula is obtained for  $f(z) = z$ .

The bracket 1.2 satisfies the Jacobi identity. In [3] we gave an indirect proof which uses the fact that there exist such coordinates that the bracket 1.2 has the standard constant form

$$\mathcal{J} = \left[ \begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array} \right]. \quad (1.3)$$

The main result of this note is a direct proof of the Jacobi identity for 1.2. In fact we give two proofs. One proof presented in Section 2 is based on direct calculations. Another proof given in Section 3 uses the language algebraic geometry. Are there

any other hierarchies of Poisson structures? We show in Section 4 that there are only two formulas of the type 1.1 corresponding to  $f(z) = 1$  and  $f(z) = z$  and that satisfy Jacobi identity. In the last Section 5 we present an example of another hierarchy of Poisson brackets and construct Darboux coordinates for it.

## 2. THE FIRST PROOF OF JACOBI IDENTITY.

First we give a proof of the Jacobi identity that use explicit form of differentials  $\epsilon_{pq}^\circ$ . Everywhere below we omit the superscript  $f$  in the formula  $\{ , \} = \{ , \}^f$ .

**Theorem 2.1.** *The bracket defined by 1.2 satisfies the Jacobi identity<sup>1</sup>*

$$\{\{w(p), w(q)\}, w(r)\} + c.p. = 0.$$

*Proof.* From the definition

$$\{w(p), w(q)\} = \frac{1}{2\pi i} \int_{\bigcup O_k} \frac{dz f(z) w(z)}{(z-p)(z-q)} \times (w(p) - w(q)).$$

Therefore,

$$\begin{aligned} \{\{w(p), w(q)\}, w(r)\} &= \frac{1}{2\pi i} \int_{\bigcup O_k} \frac{dz f(z)}{(z-p)(z-q)} \times \{w(z) (w(p) - w(q)), w(r)\} \\ &= \frac{1}{2\pi i} \int_{\bigcup O_k} \frac{dz f(z) w(z)}{(z-p)(z-q)} \times \{(w(p) - w(q)), w(r)\} + \\ &+ \frac{1}{2\pi i} \int_{\bigcup O_k} \frac{dz f(z)}{(z-p)(z-q)} \times \{w(z), w(r)\} (w(p) - w(q)) \\ &= I + II. \end{aligned}$$

---

<sup>1</sup>*c.p.* stands for cyclical permutations.

For the first term we have

$$\begin{aligned}
I &= \frac{1}{2\pi i} \int_{\bigcup O_k} \frac{dz f(z) w(z)}{(z-p)(z-q)} \times \frac{1}{2\pi i} \int_{\bigcup O_{k'}} \frac{d\eta f(\eta) w(\eta)}{(\eta-p)(\eta-r)} (w(p) - w(r)) \\
&- \frac{1}{2\pi i} \int_{\bigcup O_k} \frac{dz f(z) w(z)}{(z-p)(z-q)} \times \frac{1}{2\pi i} \int_{\bigcup O_{k'}} \frac{d\eta f(\eta) w(\eta)}{(\eta-q)(\eta-r)} (w(q) - w(r)) \\
&= \frac{1}{(2\pi i)^2} \int_{\bigcup O_k} \int_{\bigcup O_{k'}} \frac{dz d\eta f(z) f(\eta) w(z) w(\eta) (w(p) - w(r)) (z-r)(\eta-q)}{(z-p)(z-q)(z-r)(\eta-p)(\eta-r)(\eta-q)} \\
&- \frac{1}{(2\pi i)^2} \int_{\bigcup O_k} \int_{\bigcup O_{k'}} \frac{dz d\eta f(z) f(\eta) w(z) w(\eta) (w(q) - w(r)) (z-r)(\eta-p)}{(z-p)(z-q)(z-r)(\eta-p)(\eta-r)(\eta-q)}.
\end{aligned}$$

Denoting  $\mathcal{P}(z) = (z-p)(z-q)(z-r)$ ,

$$\begin{aligned}
I &= \frac{1}{(2\pi i)^2} \int_{\bigcup O_k} \int_{\bigcup O_{k'}} \frac{dz d\eta f(z) f(\eta) w(z) w(\eta) (w(p) - w(r)) (z-r)(\eta-q)}{\mathcal{P}(z) \mathcal{P}(\eta)} \\
&- \frac{1}{(2\pi i)^2} \int_{\bigcup O_k} \int_{\bigcup O_{k'}} \frac{dz d\eta f(z) f(\eta) w(z) w(\eta) (w(q) - w(r)) (z-r)(\eta-p)}{\mathcal{P}(z) \mathcal{P}(\eta)}
\end{aligned}$$

After simple algebra

$$\begin{aligned}
I + c.p. &= w(p)(q-r) \frac{1}{(2\pi i)^2} \int_{\bigcup O_k} \int_{\bigcup O_{k'}} \frac{dz d\eta f(z) f(\eta) w(z) w(\eta) (\eta+p-2z)}{\mathcal{P}(z) \mathcal{P}(\eta)} \\
&+ w(q)(r-p) \dots \\
&+ w(r)(p-q) \dots
\end{aligned}$$

Similar for the second term we have

$$\begin{aligned}
II &= \frac{1}{2\pi i} \int_{\bigcup O_{k'}} \frac{dz f(z)}{(z-p)(z-q)} (w(p) - w(q)) \times \frac{1}{2\pi i} \int_{\bigcup O_k} \frac{d\eta f(\eta) w(\eta)}{(\eta-z)(\eta-r)} (w(z) - w(r)) \\
&= \frac{1}{(2\pi i)^2} \int_{\bigcup O_{k'}} \int_{\bigcup O_k} \frac{d\eta dz f(z) f(\eta) w(\eta) w(z) (w(p) - w(q)) (\eta-p)(\eta-q)(z-r)}{\mathcal{P}(z) \mathcal{P}(\eta) (\eta-z)} \\
&- \frac{1}{(2\pi i)^2} \int_{\bigcup O_{k'}} \int_{\bigcup O_k} \frac{d\eta dz f(z) f(\eta) w(\eta) w(r) (w(p) - w(q)) (\eta-p)(\eta-q)(z-r)}{\mathcal{P}(z) \mathcal{P}(\eta) (\eta-z)} \\
&= A + B.
\end{aligned}$$

From simple algebra

$$\begin{aligned}
B + c.p. &= w(q)w(p)(p-q) \frac{1}{(2\pi i)^2} \int_{\bigcup O_{k'}} \int_{O_k} \frac{d\eta dz f(z)f(\eta)w(\eta)(\eta-r)}{\mathcal{P}(z)\mathcal{P}(\eta)} \\
&+ w(p)w(r)(r-p) \dots \\
&+ w(r)w(q)(q-r) \dots
\end{aligned}$$

Changing the order of integration

$$\int_{\bigcup O_{k'}} \int_{O_k} \frac{d\eta dz f(z)f(\eta)w(\eta)(\eta-r)}{\mathcal{P}(z)\mathcal{P}(\eta)} = \int_{\bigcup O_{k'}} \frac{d\eta f(\eta)w(\eta)(\eta-r)}{\mathcal{P}(\eta)} \int_{O_k} \frac{dz f(z)}{\mathcal{P}(z)}.$$

The differential  $dzf(z)/\mathcal{P}(z)$  is analytic inside the circles  $O_k$  and the integral vanishes due to the Cauchy theorem. Therefore,

$$B + c.p. = 0.$$

This implies

$$\begin{aligned}
II + c.p. = A + c.p. &= w(p)(q-r) \frac{1}{(2\pi i)^2} \int_{\bigcup O_k} \int_{O_{k'}} \frac{dz d\eta f(z)f(\eta)w(z)w(\eta)(\eta-p)}{\mathcal{P}(z)\mathcal{P}(\eta)} \\
&+ w(q)(r-p) \dots \\
&+ w(r)(p-q) \dots
\end{aligned}$$

Finally,

$$\begin{aligned}
I + II + c.p. &= [w(p)(q-r) + w(q)(r-p) + w(r)(p-q)] \times \\
&\times \frac{1}{(2\pi i)^2} \int_{\bigcup O_k} \int_{O_{k'}} \frac{dz d\eta f(z)f(\eta)w(z)w(\eta)(2\eta-2z)}{\mathcal{P}(z)\mathcal{P}(\eta)}.
\end{aligned}$$

The last integral vanishes due to skew symmetry.  $\square$

### 3. THE SECOND PROOF OF JACOBI IDENTITY.

Now we give a proof of the Jacobi identity that does not use explicit form of differentials  $\epsilon_{pq}^\circ$ . This proof is a reformulation of direct computations above. The proof uses the language of algebraic geometry and can be extended to general spectral curves.

*Proof.* From the definition

$$\{w(p), w(q)\} = \int_{\bigcup O_k} [\epsilon_{pq}^\circ f](z) \times w(z) (w(p) - w(q)).$$

Therefore,

$$\begin{aligned}
\{\{w(p), w(q)\}, w(r)\} &= \int_{\cup O_k} [\epsilon_{pq}^\circ f] (z) \times \{w(z) (w(p) - w(q)), w(r)\} \\
&= \int_{\cup O_k} [\epsilon_{pq}^\circ f] (z) \times w(z) \{w(p) - w(q), w(r)\} + \\
&\quad + \int_{\cup O_k} [\epsilon_{pq}^\circ f] (z) \times (w(p) - w(q)) \{w(z), w(r)\} \\
&= I + II.
\end{aligned}$$

For the first term we have

$$\begin{aligned}
I &= \int_{\cup O_k} [\epsilon_{pq}^\circ fw] (z) \times \{w(p), w(r)\} - \int_{\cup O_k} [\epsilon_{pq}^\circ fw] (z) \times \{w(q), w(r)\} \\
&= \int_{\cup O_k} [\epsilon_{pq}^\circ fw] (z) \times \int_{\cup O_{k'}} [\epsilon_{pr}^\circ fw] (\eta) \times (w(p) - w(r)) \\
&\quad - \int_{\cup O_k} [\epsilon_{pq}^\circ fw] (z) \times \int_{\cup O_{k'}} [\epsilon_{qr}^\circ fw] (\eta) \times (w(q) - w(r)) \\
&= + w(p) \int_{\cup O_k} \int_{\cup O_{k'}} [\epsilon_{pq}^\circ fw] (z) [\epsilon_{pr}^\circ fw] (\eta) \\
&\quad - w(r) \int_{\cup O_k} \int_{\cup O_{k'}} [\epsilon_{pq}^\circ fw] (z) [\epsilon_{pr}^\circ fw] (\eta) \\
&\quad + w(r) \int_{\cup O_k} \int_{\cup O_{k'}} [\epsilon_{pq}^\circ fw] (z) [\epsilon_{qr}^\circ fw] (\eta) \\
&\quad - w(q) \int_{\cup O_k} \int_{\cup O_{k'}} [\epsilon_{pq}^\circ fw] (z) [\epsilon_{qr}^\circ fw] (\eta).
\end{aligned}$$

After simple algebra

$$\begin{aligned}
I + c.p. &= w(p) \int \int [\epsilon_{pq}^\circ fw](z) [\epsilon_{pr}^\circ fw](\eta) - [\epsilon_{rp}^\circ fw](z) [\epsilon_{pq}^\circ fw](\eta) \\
&\quad - [\epsilon_{qr}^\circ fw](z) [\epsilon_{qp}^\circ fw](\eta) + [\epsilon_{qr}^\circ fw](z) [\epsilon_{rp}^\circ fw](\eta) \\
&\quad + w(q) \dots \\
&\quad + w(r) \dots
\end{aligned}$$

Using the first identity

$$\frac{\epsilon_{ab}^\circ(z)}{z-c} = \frac{\epsilon_{a'b'}^\circ(z)}{z-c'}, \quad (3.1)$$

where  $(a', b', c')$  is an arbitrary permutation of the points  $(a, b, c)$ , and the second identity

$$(z-r)(\eta-q) - (z-q)(\eta-r) - (z-p)(\eta-r) + (z-p)(\eta-q) = (\eta+p-2z)(q-r),$$

we transform the expression under integral sign to the form

$$\begin{aligned}
I + c.p. &= w(p)(q-r) \int \int \frac{[\epsilon_{pq}^\circ fw](z) [\epsilon_{pr}^\circ fw](\eta)}{(z-r)(\eta-q)} (\eta+p-2z) \\
&\quad + w(q) \dots \\
&\quad + w(r) \dots
\end{aligned}$$

Similar for the second term we have

$$\begin{aligned}
II &= (w(p) - w(q)) \times \int_{\cup O_k} [\epsilon_{pq}^\circ f](z) \int_{\cup O_{k'}} [\epsilon_{zr}^\circ fw](\eta) \times (w(z) - w(r)) \\
&= (w(p) - w(q)) \times \int_{\cup O_k} [\epsilon_{pq}^\circ fw](z) \int_{\cup O_{k'}} [\epsilon_{zr}^\circ fw](\eta) \\
&\quad - (w(p) - w(q)) w(r) \times \int_{\cup O_k} [\epsilon_{pq}^\circ f](z) \int_{\cup O_{k'}} [\epsilon_{zr}^\circ fw](\eta) \\
&= A - B.
\end{aligned}$$

It is easy to see

$$\begin{aligned}
A + c.p. &= w(p) \left[ \int_{\cup O_k} [\epsilon_{pq}^\circ fw](z) \int_{\cup O_{k'}} [\epsilon_{zr}^\circ fw](\eta) - \int_{\cup O_k} [\epsilon_{rp}^\circ f](z) \int_{\cup O_{k'}} [\epsilon_{zq}^\circ fw](\eta) \right] \\
&\quad + w(q) \dots \\
&\quad + w(r) \dots
\end{aligned}$$

Using the identity

$$(z - r)(\eta - p)(\eta - q) - (z - q)(\eta - p)(\eta - r) = (\eta - p)(r - q)(z - \eta)$$

we obtain

$$\begin{aligned} A + c.p. &= w(p)(r - q) \int_{\cup O_k} \int_{\cup O_{k'}} \frac{[\epsilon_{pq}^\circ f w](z) [\epsilon_{zr}^\circ f w](\eta)}{(z - r)(\eta - q)} (z - \eta) \\ &+ w(q) \dots \\ &+ w(r) \dots \end{aligned}$$

Using the identity

$$\frac{\epsilon_{zr}^\circ(\eta)}{\eta - q} (z - \eta) = -\epsilon_{rq}^\circ,$$

we have

$$\begin{aligned} A + c.p. &= w(p)(q - r) \int_{\cup O_k} \int_{\cup O_{k'}} \frac{[\epsilon_{pq}^\circ f w](z) [\epsilon_{rq}^\circ f w](\eta)}{(z - r)(\eta - p)} (\eta - p) \\ &+ w(q) \dots \\ &+ w(r) \dots \end{aligned}$$

From simple algebra

$$\begin{aligned} B + c.p. &= w(q)w(p) \left[ \int_{\cup O_k} [\epsilon_{qr}^\circ f](z) \int_{\cup O_{k'}} [\epsilon_{zp}^\circ f w](\eta) - \int_{\cup O_k} [\epsilon_{rp}^\circ f](z) \int_{\cup O_{k'}} [\epsilon_{zq}^\circ f w](\eta) \right] \\ &+ w(p)w(r) \dots \\ &+ w(r)w(q) \dots \end{aligned}$$

We are going to transform the expression in the square bracket using the first identity 3.1 and the second identity

$$(z - p)(\eta - q)(\eta - r) - (z - q)(\eta - p)(\eta - r) = (\eta - r)(z - \eta)(p - q).$$

Therefore,

$$\begin{aligned} B + c.p. &= w(q)w(p) \times \left[ \int_{\cup O_k} \int_{\cup O_{k'}} \frac{[\epsilon_{qr}^\circ f](z) [\epsilon_{zp}^\circ f w](\eta)(\eta - r)(z - \eta)(p - q)}{(z - p)(\eta - q)(\eta - r)} \right] \\ &+ w(p)w(r) \dots \\ &+ w(r)w(q) \dots \end{aligned}$$



Note,

$$\epsilon_{zp}^\circ(\eta)(z - \eta) = -\frac{d\eta}{\eta - p} = \epsilon_{p\infty}^\circ(\eta).$$

Changing the order of integration

$$\begin{aligned} & \int_{\bigcup O_k} \int_{\bigcup O_{k'}} \frac{[\epsilon_{qr}^\circ f](z) [\epsilon_{zp}^\circ fw](\eta)(\eta - r)(z - \eta)(p - q)}{(z - p)(\eta - q)(\eta - r)} \\ &= \int_{\bigcup O_{k'}} \frac{[\epsilon_{p\infty}^\circ fw](\eta)(\eta - r)(p - q)}{(\eta - q)(\eta - r)} \int_{\bigcup O_k} \frac{[\epsilon_{qr}^\circ f](z)}{z - p} \end{aligned}$$

The differential is analytic inside the circles  $O_k$  and the integral vanishes due to the Cauchy theorem. Therefore,

$$B + c.p. = 0.$$

This implies

$$\begin{aligned} II + c.p. = A + c.p. &= w(p)(q - r) \int_{\bigcup O_k} \int_{\bigcup O_{k'}} \frac{[\epsilon_{pq}^\circ fw](z) [\epsilon_{rq}^\circ fw](\eta)}{(z - r)(\eta - p)} (\eta - p) \\ &+ w(q)(r - p) \dots \\ &+ w(r)(p - q) \dots \end{aligned}$$

Finally,

$$\begin{aligned} I + II + c.p. &= [w(p)(q - r) + w(q)(r - p) + w(r)(p - q)] \times \\ &\times \int_{\bigcup O_k} \int_{\bigcup O_{k'}} \frac{[\epsilon_{pq}^\circ fw](z) [\epsilon_{rq}^\circ fw](\eta)}{(z - r)(\eta - p)} (2\eta - 2z). \end{aligned}$$

The last integral vanishes due to skew symmetry.  $\square$

#### 4. THE POISSON BRACKET IN TERMS OF THE WEYL FUNCTION.

By the Cauchy formula from 1.2 we have

$$\begin{aligned} \{w(p), w(q)\}^f &= \text{res}_p \alpha_{pq}^f + \text{res}_q \alpha_{pq}^f + \text{res}_\infty \alpha_{pq}^f \\ &= \frac{f(p)w(p) - f(q)w(q)}{p - q} (w(p) - w(q)) + \text{res}_\infty \alpha_{pq}^f. \end{aligned}$$

If  $f(z) = z^n$ ,  $n = 0, 1, \dots$ ; then the residue at infinity vanishes identically only for  $n = 0$  or  $1$ . The bracket in these cases  $n = 0$  or  $n = 1$  has the form

$$\{w(p), w(q)\}^1 = \frac{w(p) - w(q)}{p - q} (w(p) - w(q)),$$

and

$$\{w(p), w(q)\}^z = \frac{pw(p) - qw(q)}{p - q} (w(p) - w(q)).$$

It can be verified directly that these brackets satisfy Jacobi identity. Unfortunately these are the only examples of Poisson brackets of such form as the following theorem shows.

**Theorem 4.1.** *If  $f(z) = z^n$ ,  $n = 0, 1, \dots$ ; then the bracket defined as*

$$\{w(p), w(q)\}^f = \frac{f(p)w(p) - f(q)w(q)}{p - q} (w(p) - w(q)),$$

*satisfies the Jacobi identity only for  $n = 0$  or  $1$ .*

*Proof.* It can be verified in a lengthy but straightforward computation.

## 5. ANOTHER HIERARCHY OF POISSON BRACKETS.

The new  $n$ -th bracket is defined by the formula<sup>2</sup>

$$\{w(p), w(q)\}^n = p^n w'(p)w(q) - q^n w'(q)w(p). \quad (5.1)$$

We omit the index  $n = 0, 1, \dots$ ; for the rest of this section  $\{, \} = \{, \}^n$ .

**Theorem 5.1.** *The bracket 5.1 satisfies the Jacobi identity*

$$\{\{w(p), w(q)\}, w(r)\} + c.p. = 0.$$

*Proof.* From 5.1 we have

$$\{w'(p), w(q)\} = np^{n-1}w'(p)w(q) + p^n w''(p)w(q) - q^n w'(q)w'(p). \quad (5.2)$$

We compute the first term in Jacobi identity. Using Leibniz rule

$$\begin{aligned} \{\{w(p), w(q)\}, w(r)\} &= \{p^n w'(p)w(q), w(r)\} - \{q^n w'(q)w(p), w(r)\} \\ &= +p^n w(q)\{w'(p), w(r)\} + p^n w'(p)\{w(q), w(r)\} - \\ &\quad - q^n w(p)\{w'(q), w(r)\} - q^n w'(q)\{w(p), w(r)\}. \end{aligned}$$

Applying 5.2

$$\begin{aligned} \dots &= +p^n w(q)np^{n-1}w'(p)w(r) + p^n w(q)p^n w''(p)w(r) - p^n w(q)r^n w'(r)w'(p) \\ &\quad + p^n w'(p)q^n w'(q)w(r) - p^n w'(p)r^n w'(r)w(q) \\ &\quad - q^n w(p)nq^{n-1}w'(q)w(r) - q^n w(p)q^n w''(q)w(r) + q^n w(p)r^n w'(r)w'(q) \\ &\quad - q^n w'(q)p^n w'(p)w(r) + q^n w'(q)r^n w(p)w'(r). \end{aligned}$$

---

<sup>2</sup> The formula for  $n = 0$  was proposed by Philip de Francesco.

After a few cancellations the first term becomes

$$\begin{aligned}\{\{w(p), w(q)\}, w(r)\} &= +p^{2n}w(q)w''(p)w(r) - q^{2n}w(p)w''(q)w(r) \\ &\quad - 2p^n r^n w'(p)w'(r)w(q) + 2q^n r^n w'(q)w'(r)w(p) \\ &\quad + np^{2n-1}w(q)w'(p)w(r) - nq^{2n-1}w(p)w'(q)w(r).\end{aligned}$$

Circular permutations of variables produce

$$\begin{aligned}\{\{w(r), w(p)\}, w(q)\} &= +r^{2n}w(p)w''(r)w(q) - p^{2n}w(r)w''(p)w(q) \\ &\quad - 2r^n q^n w'(r)w'(q)w(p) + 2p^n q^n w'(p)w'(q)w(r) \\ &\quad + nr^{2n-1}w(p)w'(r)w(q) - np^{2n-1}w(r)w'(p)w(q).\end{aligned}$$

and

$$\begin{aligned}\{\{w(q), w(r)\}, w(p)\} &= +q^{2n}w(r)w''(q)w(p) - r^{2n}w(q)w''(r)w(p) \\ &\quad - 2q^n p^n w'(q)w'(p)w(r) + 2r^n p^n w'(r)w'(p)w(q) \\ &\quad + nq^{2n-1}w(r)w'(q)w(p) - nr^{2n-1}w(q)w'(r)w(p).\end{aligned}$$

The sum of three brackets is zero.  $\square$

Now we compute using methods of [4] Darboux coordinates for the bracket  $\{, \}^n$ .

**Theorem 5.2.** *For the bracket 5.1 the following identities hold*

$$\{\rho_k, \rho_p\} = -\rho_k \rho_p n(z_k^{n-1} - z_p^{n-1}), \quad (5.3)$$

$$\{\rho_p, z_k\} = \rho_p z_k^n, \quad (5.4)$$

$$\{z_k, z_p\} = 0. \quad (5.5)$$

*Proof.* Note that

$$\rho_k = \frac{1}{2\pi i} \int_{O_k} w(\zeta) d\zeta, \quad \rho_k z_k = \frac{1}{2\pi i} \int_{O_k} \zeta w(\zeta) d\zeta.$$

Therefore, integrating by parts

$$\frac{1}{2\pi i} \int_{O_k} d\zeta \zeta^n w'(\zeta) = -\frac{1}{2\pi i} \int_{O_k} d\zeta n \zeta^{n-1} w(\zeta) = -nz_k^{n-1} \rho_k.$$

To prove 5.3 we have

$$\begin{aligned}
\{\rho_k, \rho_p\} &= \left\{ \frac{1}{2\pi i} \int_{O_k} w(\zeta) d\zeta, \frac{1}{2\pi i} \int_{O_p} w(\eta) d\eta \right\} \\
&= \frac{1}{(2\pi i)^2} \int_{O_k} \int_{O_p} d\zeta d\eta [\zeta^n w'(\zeta) w(\eta) - \eta^n w'(\eta) w(\zeta)] \\
&= \frac{1}{2\pi i} \int_{O_k} d\zeta \zeta^n w'(\zeta) \frac{1}{2\pi i} \int_{O_p} d\eta w(\eta) - \frac{1}{2\pi i} \int_{O_p} d\eta \eta^n w'(\eta) \frac{1}{2\pi i} \int_{O_k} d\zeta w(\zeta) \\
&= -\rho_k n z_k^{n-1} \rho_p + \rho_k n z_p^{n-1} \rho_p.
\end{aligned}$$

To prove 5.4 we have

$$\begin{aligned}
\{z_k \rho_k, \rho_p\} &= \left\{ \frac{1}{2\pi i} \int_{O_k} \zeta w(\zeta) d\zeta, \frac{1}{2\pi i} \int_{O_p} w(\eta) d\eta \right\} \\
&= \frac{1}{(2\pi i)^2} \int_{O_k} \int_{O_p} d\zeta d\eta [\zeta^{n+1} w'(\zeta) w(\eta) - \eta^n w'(\eta) \zeta w(\zeta)] \\
&= \frac{1}{2\pi i} \int_{O_k} d\zeta \zeta^{n+1} w'(\zeta) \frac{1}{2\pi i} \int_{O_p} d\eta w(\eta) - \frac{1}{2\pi i} \int_{O_p} d\eta \eta^n w'(\eta) \frac{1}{2\pi i} \int_{O_k} d\zeta \zeta w(\zeta) \\
&= -\rho_k (n+1) z_k^n \rho_p + z_k \rho_k n z_p^{n-1} \rho_p.
\end{aligned}$$

By Leibnitz rule

$$\begin{aligned}
-\rho_k (n+1) z_k^n \rho_p + z_k \rho_k n z_p^{n-1} \rho_p &= \{z_k \rho_k, \rho_p\} = z_k \{\rho_k, \rho_p\} + \rho_k \{z_k, \rho_p\} \\
&= -n z_k \rho_k \rho_p (z_k^{n-1} - z_p^{n-1}) + \rho_k \{z_k, \rho_p\}.
\end{aligned}$$

Therefore,

$$\rho_k \{z_k, \rho_p\} = -\rho_k (n+1) z_k^n \rho_p + n \rho_k \rho_p z_k^n = -\rho_k \rho_p z_k^n.$$

To prove 5.5 we have

$$\begin{aligned}
\{z_k \rho_k, z_p \rho_p\} &= \left\{ \frac{1}{2\pi i} \int_{O_k} \zeta w(\zeta) d\zeta, \frac{1}{2\pi i} \int_{O_p} \eta w(\eta) d\eta \right\} \\
&= \frac{1}{(2\pi i)^2} \int_{O_k} \int_{O_p} d\zeta d\eta [\zeta^{n+1} w'(\zeta) \eta w(\eta) - \eta^{n+1} w'(\eta) \zeta w(\zeta)] \\
&= \frac{1}{2\pi i} \int_{O_k} d\zeta \zeta^{n+1} w'(\zeta) \frac{1}{2\pi i} \int_{O_p} d\eta \eta w(\eta) - \frac{1}{2\pi i} \int_{O_p} d\eta \eta^{n+1} w'(\eta) \frac{1}{2\pi i} \int_{O_k} d\zeta \zeta w(\zeta) \\
&= -\rho_k (n+1) z_k^n z_p \rho_p + \rho_p (n+1) z_p^n z_k \rho_k.
\end{aligned}$$

Again, by Leibnitz rule

$$\begin{aligned}
-\rho_k(n+1)z_k^n z_p \rho_p &+ \rho_p(n+1)z_p^n z_k \rho_k = \{z_k \rho_k, z_p \rho_p\} \\
&= \rho_k \rho_p \{z_k, z_p\} + \rho_k z_p \{z_k, \rho_p\} + z_k \rho_p \{\rho_k, z_p\} + z_k z_p \{\rho_k, \rho_p\} \\
&= \rho_k \rho_p \{z_k, z_p\} - \rho_k z_p z_k^n \rho_p + z_k \rho_p \rho_k z_p^n - z_k z_p \rho_k \rho_p n (z_k^{n-1} - z_p^{n-1})
\end{aligned}$$

This implies 5.5.  $\square$

Let  $n = 0$ . Then we have

$$\{\rho_k, \rho_p\} = \{z_k, z_p\} = 0, \quad \{\rho_p, z_k\} = \rho_p.$$

To reduce the bracket to constant form define

$$I_k = z_k, \quad \theta_k = \log \rho_k.$$

Then the Poisson tensor in  $I - \theta$  coordinates has the form

$$\mathcal{J} = \left[ \begin{array}{c|c} \mathbf{0} & \mathbf{1} \\ \hline -\mathbf{1} & \mathbf{0} \end{array} \right],$$

where  $\mathbf{0}$  the  $N \times N$  zero matrix and  $\mathbf{1}$  is the  $N \times N$  matrix with all entries equal to 1. The Poisson bracket is highly degenerate and has rank 2, see [7]. Let

$$\mathcal{I} = \frac{I_1 + I_2 + \dots + I_N}{N},$$

and

$$\Theta = \frac{\theta_1 + \theta_2 + \dots + \theta_N}{N}.$$

Then,

$$\{\mathcal{I}, \Theta\} = 1.$$

The matrix  $\mathbf{1}$  has rank  $N - 1$  and it is easy to construct  $2N - 2$  linear functions

$$C_1, C_2, \dots, C_{2N-2};$$

which are Casimirs of the bracket.

Let  $n = 1$ . Then we have

$$\{\rho_k, \rho_p\} = \{z_k, z_p\} = 0, \quad \{\rho_p, z_k\} = \rho_p z_k.$$

Again, let us introduce new variables

$$I_k = \log z_k, \quad \theta_k = \log \rho_k.$$

The Poisson tensor for these variables is the matrix  $\mathcal{J}$ .

For  $n = 2, 3, \dots$ ; we define

$$I_k = z_k^{-n+1}, \quad \theta_k = \log \rho_k.$$

The Poisson tensor is the matrix  $\mathcal{J}$  and the analysis above can be applied.

## REFERENCES

- [1] M. Atiyah and N. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles*. Princeton Univ. Press, (1988).
- [2] Faybusovich, L. and Gekhtman, M. *Poisson brackets on rational functions and multi-Hamiltonian structure for integrable lattices*. Phys. Lett. A 272 (2000), no. 4, 236-244.
- [3] M. Gekhtman and K.L. Vaninsky. *The family of analytic Poisson brackets for the Camassa-Holm hierarchy*. Math. Res. Lett. 15 (2008), no. 5, 867-879;
- [4] K.L. Vaninsky. *The Atiyah-Hitchin bracket and the open Toda lattice*. J. Geom. Phys. 46 (2003), no. 3-4, 283-307.
- [5] K.L. Vaninsky. *Equations of Camassa-Holm Type and Jacobi Ellipsoidal Coordinates*. Comm. Pure and Applied Math, Vol. LVIII, 1149-1187 (2005)
- [6] H. Weyl *Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlichen Funktionen*. Math Ann. 68, (1910), 220–269.
- [7] A. Weinstein *The local structure of Poisson manifolds*. J. Diff Geometry. vol. 18, 3, (1983), 523–557.

Kirill Vaninsky  
 Department of Mathematics  
 Michigan State University  
 East Lansing, MI 48824  
 USA  
 vaninsky@math.msu.edu